UNKNOTTED SOLID TORI
AND GENUS ONE WHITEHEAD MANIFOLDS

EDWARD M. BROWN

ABSTRACT. In this paper we study contractible open 3-manifolds which are monotone unions of solid tori and which embed in a compact 3-manifold. We show that the tori are unknotted in later tori. We then study pairs of unknotted solid tori, and prove a unique prime decomposition theorem. This is applied to the open 3-manifolds above to get an essentially unique prime decomposition. A number of examples in the literature are analyzed, and some new examples are constructed.

INTRODUCTION

This paper studies those open subsets of compact 3-manifolds which are contractible and have genus 1 at ∞ (see §1 for definitions). If such a subset is irreducible then it is a union of a monotone sequence of solid tori. These have been called genus one Whitehead manifolds [11, 13] in honor of J. H. C. Whitehead who exhibited the first one in 1935 in [18, 19]. Whitehead's example was described again by Bing in [1]. If W denotes Whitehead's example then Glimm and Shapiro showed $W \times \mathbb{R}^1$ is homeomorphic to $\mathbb{R}^4$, and Glimm [4] showed that $W \times W$ is homeomorphic to $\mathbb{R}^6$. McMillan and Zeeman showed that 'homeomorphic' can be replaced by 'combinatorially equivalent' in [12]. McMillan [10] extended Whitehead's construction to get an uncountable family of topologically distinct genus one Whitehead manifolds embedded in the 3-sphere S^3. In the same paper McMillan studied an example of a genus one Whitehead manifold which Bing had conjectured would not embed in S^3. Later Kister and McMillan returned to Bing's example in [8] and showed that it, and uncountably many similar examples, do not embed in S^3. In [5], Haken showed that these examples do not embed in any compact 3-manifold. Poénaru [15] proved that any solid torus in a simply connected 3-manifold is a stage in the construction of a genus one Whitehead manifold. This author [2] showed that each member of a class of 3-manifolds, which includes all genus n Whitehead manifolds, is topologically determined by its proper homotopy type. McMillan and Thickstun showed in [11] that a genus n Whitehead manifold which embeds in a compact 3-manifold having no homotopy sphere factors, also embeds in S^3. In the same paper they also point out by a cardinality argument that some genus one Whitehead manifold is not the universal cover of any compact

3-manifold, but their argument does not produce any specific example. In [13], Myers showed that a genus one Whitehead manifold is not a nontrivial cover of any 3-manifold, in particular, it is not the universal cover of any compact 3-manifold.

Our results in this paper are the following. We first make the simple observation that a contractible open set in a compact 3-manifold which has genus 1 at ∞ is the connected sum of an irreducible such set with a homotopy 3-sphere. We then observe in Lemma 4.3 that when it is irreducible such a set is a union of solid tori; that is, it is a genus one Whitehead manifold (a genus one Whitehead manifold is irreducible). Suppose that $O = \bigcup\{V_n\}$ is a genus one Whitehead manifold in a compact 3-manifold that is given as the union of the solid tori V_n. Using arguments of Haken's, we show in Theorem 4.4 that for a sufficiently large n, the pair (V_{n+1}, V_n) of solid tori has an embedding in S^3 with both solid tori unknotted (we call this an unknotted pair). Work of Schubert [16] shows that if (A, B) is a nontrivial unknotted pair, then there is a decomposition $B = E_0 \subset E_1 \subset \cdots \subset E_n = A$ into a sequence of solid tori $\{E_k\}$ so that every pair (E_{k+1}, E_k) is unknotted and prime. The heart of the paper is Theorem 3.1 which shows that the decomposition is unique up to isotopy rel B. This leads to the main Theorem, 4.6, that a genus one Whitehead manifold which embeds in a compact 3-manifold has an essentially unique prime decomposition. (The word *essentially* here refers to the fact that one can change a sequence $\{V_n\}$ by drawing an arbitrary solid torus V inside V_1 and renumbering to make it the first term.) Myers proved Theorem 4.6 in the special case that the sequence $\{V_n\}$ is what he calls excellent; that is, any essential annulus in $V_n^c \cap V_{n+1}$ is boundary parallel, and V_n is contractible in V_{n+1} (then (V_{n+1}, V_n) is prime). As Myers points out, Whitehead's example is excellent, but McMillan's examples are not. We show by example that there are uncountably many excellent genus one Whitehead manifolds. By example we also show the assumption that the genus one Whitehead manifold embeds in a compact 3-manifold is necessary for Theorem 4.6. This is because a pair (A, B) of solid tori can have distinct prime decompositions if it is not unknotted. We point out that McMillan's examples show that if one only assumes that (V_{n+1}, V_n) is prime among contractible pairs (and not among all pairs) then there are prime decompositions that have no terms in common. We also show an example of a genus one Whitehead manifold with a decomposition $\{V_n\}$ such that (V_{n+1}, V_n) is prime and unknotted for every n, but for infinitely many n's, (V_{n+2}, V_n) is a knotted pair. This manifold embeds in no compact 3-manifold. We show that a genus one Whitehead manifold which embeds in a compact 3-manifold also embeds in S^3.

The paper is organized as follows. In §1 we give the basic definitions. In §2 we show that certain pairs of solid tori, called twisted pairs, are prime. In §3 we state and prove the unique prime decomposition theorem for unknotted pairs of solid tori. In §4 we prove the theorems about open 3-manifolds, and give most of the examples.

1. DEFINITIONS

We work entirely in the P.L. category. Much of our notation is taken from [6, 7]. The symbols $\simeq$ and $\sim$ mean (P.L.) *homeomorphic to* and *homotopic to*

respectively. A *solid torus* is a space A with $S^1 \times D^2 \simeq A$ (unit circle $\times$ unit disk). If $h: S^1 \times D^2 \simeq A$ then $h(S^1 \times \{p\})$, $p \in \text{int}(D^2)$, is a *core* of A, while $h(\{p\} \times D^2)$, $p \in S^1$, is a *meridian disk*, and $h(\{p\} \times \partial D^2)$ is a *meridian* of A. Any two meridian disks are isotopic in A, and cores are isotopic rel ∂A. Any simple closed curve in ∂A which is essential in ∂A but null-homotopic in A is a meridian.

Suppose $A \subset S^3$ is a solid torus. By A^c we mean the closure of $S^3 - A$; more generally, A^c is the closure of $M - A$ where M is the largest 3-manifold so far mentioned which contains A. A *longitude* of A is a simple closed curve which is essential in ∂A, but null-homologous in A^c. A is *unknotted* provided a core of A is the unknot in S^3; equivalently, A^c is a solid torus, which is true iff a longitude of A bounds a disk in A^c. Thus, if A is unknotted, a longitude of A is a meridian of A^c.

(A, B) is a *pair* of solid tori if A and B are solid tori and B is contained in the interior of A (we write $B \subset A$). A pair (A, B) of solid tori is *unknotted* if there exists an embedding $h: A \to S^3$ with $h(A)$ and $h(B)$ both unknotted. In this case we will frequently identify (A, B) with $(h(A), h(B))$ and think of (A, B) as embedded in S^3. Associated to a pair (A, B) of solid tori there are two integers, the *winding number*, $W(A, B)$, and the *wrapping number*, $N(A, B)$. The winding number is the nonnegative coefficient of a core of B in the fundamental group $\pi_1(A)$. This is zero iff B is contractible in A. The wrapping number $N(A, B)$ is the minimum number of points in which any core of B meets a meridian disk of A (cf. [16]). These numbers depend only on the isotopy class of B in A. The following is an easy remark.

Remark 1.1. Suppose (A, B) is an unknotted pair. Then $N(A, B) = 0$ iff B is contained in a cell in A and $N(A, B) = 1$ iff a core of B is also a core of A. In each case there is, up to isotopy of B in A, exactly one such pair (A, B).

An unknotted pair (A, B) is called a *trivial* pair if $N(A, B) < 2$.

Remark 1.2. Suppoe $A \supset B \supset C$ are solid tori. Then

$$W(A, C) = W(A, B) * W(B, C) \quad \text{and} \quad N(A, C) = N(A, B) * N(B, C).$$

Proof. The first equation is obvious, the second is in [16].

A nontrivial unknotted pair (A, C) is said to *factor* if there exists a solid torus B with $A \supset B \supset C$, so that (A, B) and (B, C) are both nontrivial unknotted pairs. Otherwise we say that (A, C) is a *prime pair*. By Remark 1.2, if $N(A, C)$ is a prime integer, then the unknotted pair (A, C) is a prime pair. In particular, any nontrivial pair (A, C) has a finite prime factorization.

The next theorem is mainly due to Schubert [16] and shows that the pairs (A, B) and (B, C) above are automatically unknotted.

Theorem 1.3. *Let (A, C) be an unknotted pair with $N(A, C) > 0$. Suppose $T \subset A$ is a torus which separates ∂A from ∂C. Then T bounds a solid torus B in A so that (A, B) and (B, C) are both unknotted.*

Proof. Embed (A, C) in S^3 so that both are unknotted, and let B be the closure of the component of $S^3 - T$ which contains C. Suppose, first, that B is a solid torus. Choose a meridian disk D of C^c so that D meets T in general position and in a minimal number of components (which are simple

closed curves). Any component of $D \cap T$ which is inessential on T bounds disks on both D and T. But such components can be removed by the usual disk swapping arguments, so all components of $D \cap T$ are essential on T. Let γ be a component of $D \cap T$ which bounds an innermost disk D_1 on D. If $D_1 \subset B$ then D_1 is a meridian disk of B since γ is essential on T. But a meridian disk of B must meet ∂D, for otherwise C is contained in a cell in B (and hence in A). We conclude that $D_1 \subset B^c$, so B is unknotted in S^3.

Finally, suppose B^c is a solid torus. Then making the same argument with respect to (C^c, A^c) we conclude that B^c, and hence B, is unknotted.

It follows from the above that an unknotted nontrivial pair (A, C) is prime iff every essential torus in $A \cap C^c$ is boundary parallel ($A \cap C^c$ is sometimes called *simple*).

If O is an open connected 3-manifold then $O = \bigcup \{C_n\}$ where each C_n is a compact 3-manifold, and C_n is contained in the interior of C_{n+1}. The sequence $\{C_n\}$ has been called an *exhausting sequence*, or a *defining sequence* for O. If we can choose $\{C_n\}$ with the genus of ∂C_n bounded, then we call the smallest bound for the genus of ∂C_n the *genus of O at ∞*. Notice that if the genus of O at ∞ is g, we can choose $\{C_n\}$ so that no C_n is contained in a compact submanifold C of O with the genus of ∂C less than g.

2. TWISTED PAIRS

An unknotted pair (A, B) is called a *twisted pair* if a core of B is parallel to a curve on ∂A. This means there is an annulus α embedded in A so that $\alpha \cap \partial A$ is one boundary component of α and the other boundary component of α is a core of B. The main result of this section is that a twisted nontrivial pair is prime.

In what follows we shall often speak of a *meridian-longitude pair* for a solid torus A embedded in S^3. This means a pair (m, l) consisting of a meridian m and a longitude l for A which intersect in one point. We give A the orientation induced from S^3 and ∂A the orientation induced from A. We consider the curves m and l to be oriented so that (m, l), in that order, induces the positive orientation on ∂A. Note that $(-m, -l)$ is also a meridian-longitude pair, but $(-m, l)$ and $(m, -l)$ are not.

Lemma 2.1. *Let* (A, B) *be a twisted nontrivial pair considered as embedded in* S^3. *Let* b *be a curve on* ∂A *that is parallel to a core of* B. *Then for an appropriately oriented meridian-longitude pair* (m, l) *of* A *we have* $b \sim k * l \pm m$, *where* $k = W(A, B) = N(A, B)$.

Proof. The curve b is nonsingular on ∂A, and so $b \sim k * l + j * m$ for relatively prime integers $k \geq 0$ and j. Neither k nor j is zero since (A, B) is nontrivial, and $k \neq 1$ for the same reason. If neither k nor j were ± 1, then b would be a torus knot (see [3]) but a core of B is not parallel to a knot in S^3. Thus $j = \pm 1$ as desired.

Clearly $W(A, B) = k$, but a curve such as b meets some meridian disk in k points. It follows that there is a meridian disk of A which meets a core of B in k points so $N(A, B) = k$.

Lemma 2.2. *Let* (A, B) *be a nontrivial pair, unknotted in* S^3, *and let* (m, l), (m', l') *be meridian-longitude pairs for* A *and* B *respectively. Then the pair*

(A, B) *is a twisted pair iff there is a properly embedded annulus* $\alpha \subset A \cap B^c$ *with* $\alpha \cap \partial A \sim k * l \pm m$, *and* $\alpha \cap \partial B \sim k * m' \pm l'$.

Proof. If α exists then $\alpha \cap \partial B$ is parallel in B to a core of B, so one implication is clear.

Suppose (A, B) is a twisted pair and γ is an annulus with one boundary a core of B and the other the curve $k * l \pm m$ on ∂A. Suppose further that γ is chosen to have general position with respect to ∂B, and so that $\gamma \cap \partial B$ has a minimal number of components. A component of $\gamma \cap \partial B$ is inessential on ∂B if and only if it is inessential on γ. For $\pi_1(\gamma) \to \pi_1(A)$ is a monomorphism, so an essential curve on γ is essential in A, and hence on ∂B. If a component of $\gamma \cap \partial B$ is inessential in γ, we can select one that bounds an innermost disk D. Suppose ∂D is essential on ∂B. If $D \subset B$, then D is a meridian disk of B which misses a core of B. If $D \subset B^c$ then B is contained in a cell in A, but both are false. Any component of $\gamma \cap \partial B$ which is inessential on both ∂B and γ can be removed by disk swapping, so none exist.

The set $\gamma \cap \partial B$ has only one component since any two would be parallel on ∂B contradicting the minimal number. Let $\alpha = \gamma \cap B^c$, a subannulus of γ. Now $\alpha \cap \partial A = \gamma \cap \partial A$ which says that one boundary component of α is as desired. The other boundary, $\alpha \cap \partial B$, is parallel in B to a core of B. It follows that $\alpha \cap \partial B \sim k' * m' \pm l'$. If (m, l) is a meridian-longitude pair for A, then (l, m) is a meridian-longitude pair for A^c. Thus $\alpha \cap \partial A$ is parallel to a core of A^c, which shows that (B^c, A^c) is also a twisted pair. Finally, k and k' are both the linking number of a core of B with a core of A^c, and so are equal.

Corollary 2.3. *A pair* (A, B) *is a nontrivial twisted pair iff* (B^c, A^c) *is a nontrivial twisted pair.*

Lemma 2.4. *Suppose* $A \supset B \supset C$ *is a nontrivial factoring of the twisted pair* (A, C). *Then* (A, B) *and* (B, C) *are also twisted pairs.*

Proof. Let α be an annulus joining a core of C to a curve on ∂A. Suppose α is chosen to have general position with respect to ∂B, and so that $\alpha \cap \partial B$ has a minimal number of components. Since $\pi_1(\alpha) \to \pi_1(A)$ is a monomorphism, if a component of $\alpha \cap \partial B$ is inessential on ∂B, then it is inessential on α as well. On the other hand, if there are any components of $\alpha \cap \partial B$ which are inessential on α, we may choose an innermost one b bounding a disk $D \subset \alpha$. If b is essential on ∂B and $D \subset B$, then D is a meridian disk of B which misses a core of C. But (B, C) was assumed nontrivial. Next, if $D \subset B^c$, then a regular neighborhood of $B \cup D$ is a cell in A containing B. We conclude that b is inessential on ∂B. But then disk swapping eliminates b. Thus all components of $\alpha \cap \partial B$ are essential on both α and on ∂B. Since ∂B separates ∂A from ∂C, $\alpha \cap \partial B \neq \varnothing$. It follows that there is a subannulus of α, contained in B, bounded by a core of C and a curve on ∂B. We conclude that (B, C) is a twisted pair.

If (A, C) is a twisted pair then by Corollary 2.3, (C^c, A^c) is a twisted pair and $C^c \supset B^c \supset A^c$ is a nontrivial factorization. As above we conclude that (B^c, A^c) is a twisted pair, and, again by Corollary 2.3, that (A, B) is also a twisted pair.

Theorem 2.5. *If* (A, C) *is a nontrivial twisted pair, then* (A, C) *is a prime pair.*

Proof. Suppose to the contrary that $A \supset B \supset C$ is a nontrivial factorization with $W(A, B) = k$ and $W(B, C) = r$. Let α be an annulus in $A \cap C^c$ with $\alpha \cap \partial A \sim k * r * l \pm m$ and $\alpha \cap \partial C \sim k * r * m'' \pm l''$, where (m, l) and (m'', l'') are meridian-longitude pairs for A and C respectively (Lemma 2.2). We assume α is chosen so that it has general position and minimal intersection with ∂B. As usual, α meets ∂B in curves essential on both α and ∂B. Since these curves are parallel on ∂B there is only one of them, call it b. Now α can be extended slightly inside C to a core, so by Lemma 2.1 $b \sim r * l' \pm m'$, where (m', l') is a meridian-longitude pair for B. Similarly, α can be extended a little inside A^c to a core of A^c. Again by Lemma 2.1 applied to (B^c, A^c), $b \sim k * m' \pm l'$. It follows that $k = r = 1$, which contradicts nontriviality.

3. UNIQUE PRIME DECOMPOSITION

By a *prime decomposition* of a nontrivial unknotted pair (A, B) we mean a sequence $B = E_0 \subset E_1 \subset \cdots \subset E_k = A$, where each pair (E_{t+1}, E_t) is prime. As noted in the introduction, a prime decomposition exists for any nontrivial pair (A, B), our main theorem is uniqueness.

Theorem 3.1. *Let $B = E_0 \subset E_1 \subset \cdots \subset E_k = A$ and $B = E_0' \subset E_1' \subset \cdots \subset E_l' = A$ be two prime decompositions of the nontrivial pair (A, B), unknotted in S^3. Then $k = l$ and there is an isotopy of A, fixing points of $\partial A \cup B$, of the identity to a homeomorphism which carries E_r to E_r' for all r.*

Proof. If the theorem is false, select a counterexample so that k and then l are as small as possible. Let $T_r = \partial E_r$ and $T_r' = \partial E_r'$. By an isotopy fixing $B \cup \partial A$ move the E_r so that T_r meets T_t' in general position for all r and t, and so that the number of path components of $T_r \cap T_t'$, $\#\pi_0(T_r \cap T_t')$, is as small as possible for all r and t.

(1) *The inclusion maps induce monomorphisms $\pi_1(T_r) \to \pi_1(A \cap B^c)$, and $\pi_1(T_r') \to \pi_1(A \cap B^c)$.* Otherwise, B is contained in a cell in A.

(2) *Components of $T_r \cap T_t'$ are essential on both T_r and T_t'.* Any inessential components bound disks on both T_r and on T_t' by (1). Then using disk swapping we can decrease $\#\pi_0(T_r \cap T_t')$.

(3) *Components of $T_r \cap \bigcup_t T_t'$ are parallel on T_r, and components of $T_t' \cap \bigcup_r T_r$ are parallel on T_t',* since they are pairwise disjoint essential curves.

Suppose $T_m \cap T_n' \neq \varnothing$. Choose a meridian disk D_m of E_m so that D_m meets T_n' in general position and $\#\pi_0(D_m \cap T_n')$ is as small as possible. The rest of the proof breaks into two cases.

Case I. For some m, n, and D_m, some component of $D_m \cap T_n'$ is a simple closed curve. (Case II will be that every such component is an arc.)

(4) *Every simple closed curve component of $D_m \cap T_n'$ is essential on T_n'.* Otherwise the usual disk swapping argument will decrease $\#\pi_0(D_m \cap T_n')$.

(5) *The solid tori E_m and E_n' have a common meridian disk.* Let c be a component of $D_m \cap T_n'$ which bounds a disk D' in D_m that contains no further points of T_n' (i.e., c is innermost on D_m). Since c is essential on T_n' by (4), and E_n' is not contained in a cell in A, $D' \subset E_n'$. Thus D' is a meridian disk of E_n'. There is an annulus component α of $T_n' \cap E_m$ which contains c, and c is essential on α. Let α' be one of the two subannuli into which c cuts α. Then $\alpha' \cup D'$ is a meridian disk of E_m. Having the same boundary, but slightly to one side, is a common meridian disk of E_m and E_n'.

(6) *If E_m and E'_n have a common meridian disk D, then there is an isotopy keeping $\partial A \cup B$ pointwise fixed and moving E_m onto E'_n.* If $E_m = E'_n = A$, or $E_m = E'_n = B$ then there is nothing to prove. We assume $0 < m < k$ and $0 < n < l$. Now $c = \partial D$ is a component of $T_m \cap T'_n$. By (3) components of $E_m \cap T'_n$ are annuli bounded by meridians of both E_m and E'_n as are components of $T_m \cap E'_n$. It follows that the component F of $E_m \cap E'_n$ containing B is bounded by a torus separating ∂B from ∂A. Thus F is an unknotted solid torus. Since $D \subset E_m \cap E'_n$ and ∂D is essential on ∂F, D is a meridian disk of F. Shrinking F inside itself a little we have $N(E_m, F) = N(E'_n, F) = 1$, so there are isotopies of the desired kind moving E_m to F and F to E'_n.

(7) *In Case* I, $E_m = E'_n = A$ *or* $E_m = E'_n = B$. Otherwise (6) gives a contradiction to the choice of k and l.

This finishes Case I.

Case II. If $0 < m < k$ and $T'_n \cap T_m \neq \varnothing$, then every component of $D_m \cap T'_n$ is an arc with endpoints in ∂D_m. Note that $T'_n \cap D_m \neq \varnothing$, for otherwise E'_n and E_m have a common meridian disk and (6) gives a contradiction.

(8) $T_1 \cap T'_1 \neq \varnothing$. Otherwise, either $T'_1 \subset E_1$ or $T_1 \subset E'_1$. In the first case we have $E_0 \subset E'_1 \subset E_1$. Since (E_1, E_0) is prime $N(E_1, E'_1) = 1$, so $E_1 \cap E'^c_1 \simeq T_1 \times [0, 1]$. Then the isotopy of the required kind can be performed in a small neighborhood of $E_1 \cap E'^c_1$ moving E_1 onto E'_1. This contradicts the choice of k and l. The proof is similar in the case $T_1 \subset E'_1$.

The components of $T'_1 \cap (\bigcup_r T_r)$ cut T'_1 into annuli denoted $\{\alpha'_{1,t}\}$. Since we are in Case II, $T'_1 \cap D_1$ is a family of arcs each being a component of the intersection of an annulus $\alpha'_{1,t}$ with D_1. The endpoints of these arcs are in both ∂D_1 and $\partial \alpha'_{1,t}$ and they cut D_1 into a family of subdisks. One of these subdisks is called *outermost* if its boundary consists of a single arc on ∂D_1 and a single arc of $\alpha'_{1,t} \cap D_1$. Clearly there are outermost subdisks.

(9) *If $\alpha'_{1,t} \cap D_1 \neq \varnothing$ then each component spans $\alpha'_{1,t}$ (i.e., meets both boundary components of $\alpha'_{1,t}$).* Suppose some component of $\alpha'_{1,t} \cap D_1$ has both endpoints on the same component of $\partial \alpha'_{1,t}$. Then it cuts off a disk D' from $\alpha'_{1,t}$ and we may choose D' so that $D' \cap D_1 \subset \partial D_1$ (i.e., D' is outermost on $\alpha'_{1,t}$). The arc $D' \cap D_1$ cuts D_1 into two subdisks. We can swap D' with either of these subdisks to change D_1 to another embedded disk in E_1 since D' is outermost. One of these disks will be a meridian disk of E_1 which, after a small displacement, decreases $\#\pi_0(D_1 \cap T'_1)$.

(10) *Suppose D is an outermost subdisk of D_1 and $\alpha' = \alpha'_{1,t}$ is the annulus on T'_1 that cuts off D. Then there is an annulus $\alpha \subset T_1$ so that $\partial \alpha = \partial \alpha'$ and $\alpha \cap T'_1 = \partial \alpha$.* The boundary $\partial \alpha'$ separates T_1 into two annuli, let α be the one that contains the arc $a = \partial D \cap T_1$. Since D is outermost, $a \cap T'_1 = \partial a = a \cap \alpha'$. But a spans α' by (9) and so spans α. A spanning arc of α meets any essential curves on α, thus $\alpha \cap T'_1 = \partial \alpha$.

Fix a pair α, α' as in (10).

(11) *The arcs $\alpha \cap \partial D$ and $\alpha' \cap D$ are spanning arcs of α and α' respectively.* See (10).

(12) *If D is outermost on D_1 then $D \cap E_0 \neq \varnothing$.* Otherwise, pushing through the disk D, we can change α from an annulus to a disk, and then remove it completely. This decreases $\#\pi_0(T_1 \cap T'_1)$.

Let $T = \alpha \cup \alpha'$, a torus, and let F be the closure of the component of $S^3 - T$ containing D (and also E_0 by (12)). By Theorem 1.3, F is an unknotted solid torus. Since $E_0 \subset F \subset E_1$, either $N(F, E_0) = 1$ or $N(E_1, F) = 1$ and $N(E_1', F) = 1$.

(13) $N(F, E_0) = 1$. Otherwise shrink F slightly inside itself using a regular neighborhood of the boundary. Then $N(E_1, F) = N(E_1', F) = 1$. But the same contradiction as in (8) shows this cannot happen.

(14) *The pairs (E_1, E_0) and (E_1', E_0) are twisted pairs.* The disk D is a meridian disk of F, so a component b of $\partial \alpha$ is parallel to a core of F. By (13) a core of E_0 is a core of F, and is thus parallel in F to b. Since $b \subset T_1 \cap T_1'$, both pairs above are twisted pairs.

(15) $T_1' \cap T_2 = \varnothing$. Otherwise one of the annuli $\alpha_{1,t}'$ lies in $E_2 \cap E_1^c$ and has one boundary on T_1 and the other on T_2. But the component of $\partial \alpha_{1,t}'$ lying on T_1 is parallel on T_1 to b (see (14)). Then a core of E_0 is parallel to a curve on T_2 and the pair (E_2, E_0) is a twisted pair. This contradicts Theorem 2.5 which says that a twisted pair is prime.

(16) $k = l = 2$, *and the prime decompositions of our counterexample are* $E_0 \subset E_1 \subset E_2$, *and* $E_0 \subset E_1' \subset E_2$. By (15) $E_1' \subset E_2$ and is part of a prime decomposition of (E_2, E_0). Thus $k = 2$ and $E_0 \subset E_1' \subset \cdots \subset E_1' = E_2$. The argument repeated with the roles of E_m and E_m' interchanged shows that $l = 2$ as well.

We can apply the above arguments to the prime decompositions $E_2^c \subset E_1^c \subset E_0^c$, and $E_2^c \subset E_1'^c \subset E_0^c$. We conclude that

(17) *The pair (E_1^c, E_2^c) is a twisted pair with a core of E_2^c parallel to the same b as a core of E_0*. Both cores are parallel to a component of $T_1 \cap T_1'$.

Now let (m, l) be a meridian-longitude pair for E_1. Since (E_1, E_0) is a twisted pair by (14), Lemma 2.1 says that $b \sim k * l \pm m$, where $k = N(E_1, E_0)$. Since (l, m) is a meridian-longitude pair for E_1^c, (17) says that $b \sim k' * m \pm l$. Thus $k = k' = 1$ which contradicts the fact that the pair (E_1, E_0) is nontrivial. This contradiction finishes the proof.

4. Contractible open 3-manifolds

In this section O (or O', etc.) will always be a contractible open set of genus 1 at ∞ in a compact 3-manifold M (or M', etc.). The sequence $\{V_n\}$ will be an exhausting (or defining) sequence for O such that ∂V_n has genus 1, no V_n is contained in a compact submanifold of O with boundary of genus O, and ∂V_n is not parallel to ∂V_{n+1}. It follows from the loop theorem of [17] that $\pi_1(\partial V_n) \to \pi_1(V_0^c \cap V_n)$ and $\pi_1(\partial V_n) \to \pi_1(V_n^c \cap O)$ are both monomorphisms (here V_n^c is with respect to M). Then by [14] or [9], $\pi_1(\partial V_n) \to \pi_1(V_0^c \cap O)$ is also a monomorphism.

Theorem 4.1. *If O (as above) is not irreducible then there is a 2-sphere $\Omega \subset O$ which separates M and gives a decomposition $O \simeq O' \# S$ where S is a nontrivial homotopy 3-sphere and O' is an irreducible contractible open 3-manifold of genus 1 at ∞. Also $M \simeq M' \# S$ where M' is compact and $O' \subset M'$.*

Proof. If O is not irreducible there is a 2-sphere $\Omega_1 \subset O$ which does not bound a ball in O. Since O is contractible, Ω_1 separates O. Since O has only one end (being contractible), $\Omega_1 = \partial A_1$ for some compact $A_1 \subset O$. Then $\pi_1(A_1) \to \pi_1(M)$ is a monomorphism and factors through $\pi_1(O)$. If

S_1 is A_1 capped off with a cell, then S_1 is a nontrivial homotopy sphere and $O \simeq O_1' \# S_1$, with $M \simeq M_1' \# S_1$ and $O_1 \subset M_1$. If O_1 is not irreducible we can find a 2-sphere Ω_2 in O_1 bounding a homotopy cell A_2 containing Ω_1. Again let S_2 be A_2 capped, so $M \simeq M_2' \# S_1 \# S_2$, and $O \simeq O_2' \# S_1 \# S_2$. Since M has a finite prime decomposition this must stop after some number k of steps and $S = S_1 \# \cdots \# S_k$, $\Omega = \partial A_k'$, $O' = O_k'$, and $M' = M_k'$ are as desired.

Lemma 4.2. *If O (as above) is irreducible then each V_n is a solid torus.*

Proof. Since $\pi_1(\partial V_n) \to \pi_1(V_n^c \cap O)$ is a monomorphism, $\pi_1(\partial V_n) \to \pi_1(V_n)$ has a nontrivial kernel. The loop theorem provides a properly embedded disk D in V_n with ∂D essential in ∂V_n. Since O is contractible it is orientable and so is V_n. Since O is irreducible, cutting V_n over D gives a cell, so V_n is a solid torus.

Lemma 4.3. *For sufficiently large n, $\pi_1(\partial V_n) \to \pi_1(V_n^c)$ has a nontrivial kernel.*

Proof. By the Haken finiteness theorem [5], [6], or [7] the compact 3-manifold V_0^c does not contain an infinite number of nonparallel, incompressible tori. It follows that all but a finite number of the ∂V_n are compressible in V_0^c. Since ∂V_n is incompressible in $V_n \cap V_0^c$, ∂V_n is compressible in V_n^c if it is compressible in V_0^c.

From now on, in addition to our other assumptions, we assume that O is irreducible, that every V_n is a solid torus, and that ∂V_n is compressible in V_n^c for every n.

In [1], Bing conjectured that if V_n has a local knot in V_{n+1} (i.e., meets some cell in a single knotted tube) then O does not embed in S^3. In [8], Kister and McMillan proved Bing's conjecture. Haken [5] extended this to show that O does not even embed in a compact 3-manifold.

By a *longitude* of a solid torus A we mean a curve which is carried to a longitude by some unknotted embedding of A in S^3. By a *longitude pair* for an unknotted pair (A, B) we mean a pair of curves on $(\partial A, \partial B)$ which are carried to longitudes by an unknotted embedding of the pair (A, B) in S^3.

Theorem 4.4. *For O and $\{V_n\}$ as above, there is a longitude l_n of V_n for each n so that for every n and k, (V_{n+k}, V_n) is an unknotted pair, and (l_{n+k}, l_n) is a longitude pair.*

Proof. Let $K_n = V_n \cup U(D_n)$ where D_n is a compressing disk for ∂V_n in V_n^c. Here $U(D_n)$ is a regular neighborhood of D_n in V_n^c so that $U(D_n) \cap \partial V_n$ is a regular neighborhood of ∂D_n in ∂V_n (i.e., an annulus). I claim that K_n is a cell. The argument below follows Haken's argument [5, p. 68].

To see that K_n is a cell, choose a meridian loop m_n on ∂V_n which meets ∂D_n in general position and in a minimal number r of points. We show that $r = 1$. Let l be a simple loop on ∂V_n which meets m_n transversally in a single point. Then l generates $H_1(V_n)$ and l, m_n together generate $H_1(\partial V_n)$. As r is minimal, with appropriately chosen orientations, ∂D_n is homologous to $r * l + s * m_n$ on ∂V_n. Moreover ∂D_n is a simple closed curve, so r and s are relatively prime. It follows that $\pi_1(K_n)$ is isomorphic to Z/rZ (the integers mod r) generated by l. Since $U(D_n) \cap \partial V_n$ is an annulus, ∂K_n is a 2-sphere. Then $\pi_1(M)$ is the free product of $\pi_1(K_n)$ with $\pi_1(K_n^c)$, so $\pi_1(K_n) \to \pi_1(M)$ is a monomorphism. But the generator l of $\pi_1(K_n)$ lies in V_n and hence in O, so it is null homotopic in M. It follows that $r = 1$, and that K_n is a cell.

Let $l_n = \partial D_n$ for each n. Since l_n is inessential in V_n^c, it is inessential in $V_n^c \cap K_{n+k}$. Thus the pair (V_{n+k}, V_n) is unknotted in the cell K_{n+k}, and (l_{n+k}, l_n) is a pair of longitudes.

In [10] McMillan gave a construction that showed:

Theorem 4.5. *Suppose $\{(A_n, B_n)\}$ is a sequence of unknotted pairs. There is an open subset $O \subset S^3$, and an exhausting sequence of solid tori $\{V_n\}$ for O, so that $(V_{n+1}, V_n) \simeq (A_n, B_n)$ for every n. If there are arbitrarily large n for which B_n is contractible in A_n, then O is contractible.*

By Theorem 4.4, O has an exhausting sequence $\{V_n\}$ of solid tori so that (V_{n+1}, V_n) is unknotted for every n. If (V_{n+1}, V_n) is prime for every n, we call the sequence a *prime* sequence. As we have noted, every such O has a prime sequence. The next theorem says that essentially there is only one prime sequence for O.

Theorem 4.6. *Suppose O and O' are as above, with prime sequences $\{V_n\}$ and $\{V_n'\}$. If $O \simeq O'$ then there are integers k and l, and a homeomorphism $h: O \to O'$ so that $h(V_{n+k}) = V_{n+l}'$ for every n.*

Proof. As noted before, $\{V_n\}$ can be extended by adding arbitrary terms at the beginning so the integers k and l cannot be eliminated.

Let $h': O \to O'$ be a homeomorphism, and choose integers k and m so that $V_1' \subset h'(V_k) \subset V_m'$. By Theorem 3.1 applied to the pair (V_{m+1}', V_0'), there is an isotopy of O', fixed on $V_0' \cup V_{m+1}'^c$ carrying $h'(V_k)$ to V_l' for some l. We let h_0 be the composition of h' with the end of this isotopy.

Suppose, for some $n \geq 0$, we have constructed a homeomorphism h_n of O to O' with $h_n(V_{k+t}) = V_{l+t}'$ for all $0 \leq t \leq n$. Choose m with $V_{l+n}' \subset h_n(V_{k+n+1}) \subset V_m'$. Apply Theorem 3.1 to (V_{m+1}', V_{l+n}') to get an isotopy, fixed on $V_{l+n}' \cup V_{m+1}'^c$, and moving $h_n(V_{k+n+1})$ onto V_{l+n+1}'. We let h_{n+1} be the composition of h_n with the end of this isotopy. The desired homeomorphism is the limit of the sequence $\{h_n\}$.

We would now like to show that every O (as above) can be embedded in S^3. This was proved in [11] in the case that M is irreducible and orientable.

Lemma 4.7. *If (V_{n+1}, V_n) is an unknotted pair with longitude pair (l_{n+1}, l_n), then any unknotted embedding $f_n: V_n \to S^3$ with $f_n(l_n)$ a longitude, extends to an unknotted embedding $f_{n+1}: V_{n+1} \to S^3$ with $f_{n+1}(l_{n+1})$ a longitude.*

Proof. Let h be an unknotted embedding of (V_{n+1}, V_n) in S^3 which carries the pair (l_{n+1}, l_n) to longitudes. We may even assume that $f_n(V_n) = h(V_n)$ and $f_n(l_n) = h(l_n)$ by composing h with a homeomorphism of S^3. Now $f_n \circ h^{-1}: h(V_n) \to h(V_n)$ and maps a longitude to itself. It follows that $f_n \circ h^{-1}$ extends to a homeomorphism $H: S^3 \to S^3$. We claim that $f_{n+1} = H \circ h$ is as desired.

Clearly, $f_{n+1}|V_n = H \circ h|V_n = f_n \circ h^{-1} \circ h|V_n = f_n$. Since $h(l_{n+1})$ is a longitude of $h(V_{n+1})$, and H is defined on all of S^3, it follows that $f_{n+1}(l_{n+1})$ is a longitude of $f_{n+1}(V_{n+1})$.

Corollary 4.8. *With notation as in Theorem 4.4, there is an embedding $f: O \to S^3$ with $f(l_n)$ a longitude of $f(V_n)$ for every n.*

Consider now the examples of genus one Whitehead manifolds constructed by McMillan [10, Theorem 1]. They are all of this form: Select a sequence

$\{n_k\}$ of integers, all greater than 1. For each k let $A_k \supset B_k \supset C_k$ be unknotted solid tori in S^3 so that (A_k, B_k) is a twisted pair with $N(A_k, B_k) = n_k$ and (B_k, C_k) is the Whitehead pair. Then $\{V_k\}$ with $(V_{2k+2}, V_{2k+1}) \simeq (A_k, B_k)$ and $(V_{2k+1}, V_{2k}) \simeq (B_k, C_k)$ is a prime sequence for a contractible open set in S^3. By Theorem 4.6, two sequences $\{n_k\}$ and $\{n_k'\}$ give nonhomeomorphic open sets unless they are cofinally equal. Notice that the pairs (V_{2k+3}, V_{2k+1}) and (V_{2k+2}, V_{2k}) are prime among contractible pairs. This gives decompositions $\{V_{2k+1}\}$ and $\{V_{2k}\}$ which are prime sequences among contractible pairs for the same open set, but have no terms in common. Thus Theorem 4.6 is false if "prime" is replaced with "prime among contractible pairs".

Here is a list of infinitely many prime torus pairs (all of which have wrapping number 2) from which many more examples can be constructed (see Figure 1). In an unknotted solid torus $A \subset S^3$ select a cell in the form $D^2 \times [0, 1]$. In the cell select an arc $a \subset \text{int}(D^2)$ and a ribbon going from $a \times 0$ to $a \times 1$ which ties a knot k, and which makes n 1/2-twists (say c.c.w. if $n > 0$, and c.w. if $n < 0$). The edge of the ribbon is a pair of wires in $D^2 \times [0, 1]$. Connect the ends of the wires in $D^2 \times 1$ with an arc running around A, and similarly for the ends in $D^2 \times 0$. A regular neighborhood gives an unknotted solid torus B in A with wrapping number 2. The winding number is 0 if n is even, and 2 if n is odd. By examining the inverse image of B in the 2-fold cyclic cover of A, one can see that the homeomorphism type of (A, B) changes if k or n is changed. The assumption that the pair (A, B) is unknotted cannot be dropped from Theorem 3.1. We can get examples of this using the fact that knot multiplication is commutative. Let (A, C) be the pair so that $N(A, C) = 1$, and so that C ties two local prime knots, say a figure eight and a trefoil, as it circles A (see Figure 2). There are two prime decompositions of (A, C): $A \supset B \supset C$ where
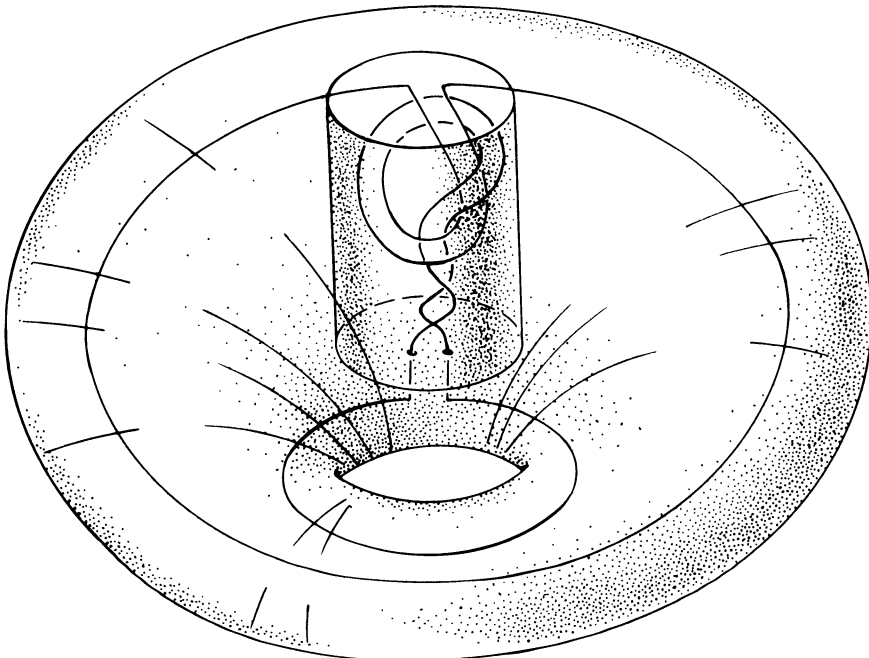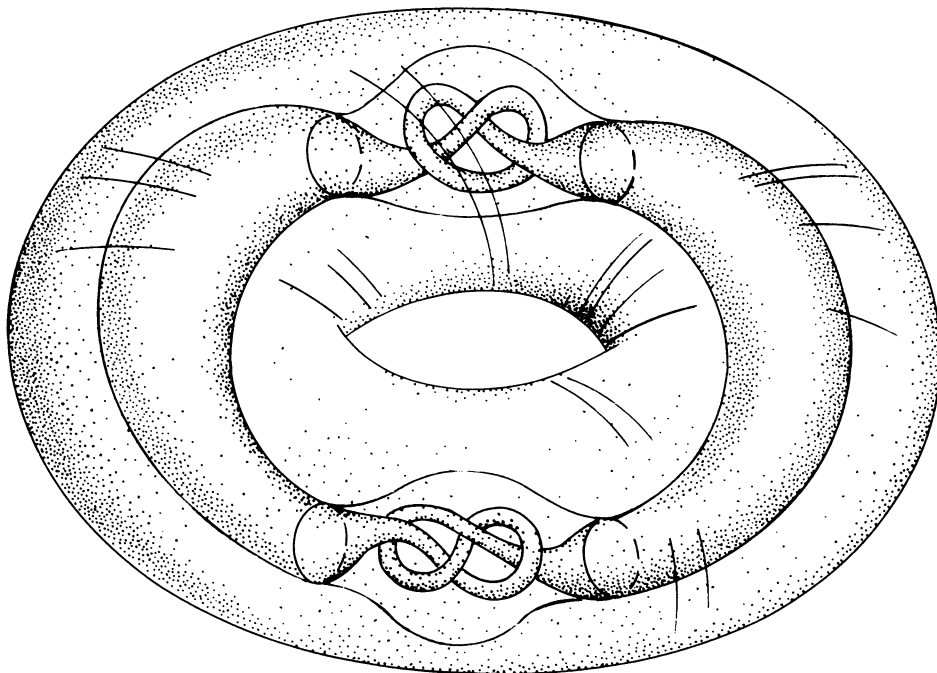

FIGURE 1

FIGURE 2

B ties a figure eight in A, and C ties a trefoil in B; and $A \supset B' \supset C$ where B' ties a trefoil in A, and C ties a figure eight in B'.

For our last example we need:

Lemma 4.9. *Suppose* (A, B) *is a twisted pair with* $N(A, B) = k$, $k \geq 3$. *If* $f, g: (A, B) \to S^3$ *are unknotted embeddings, then* gf^{-1} *extends to a homeomorphism of* S^3 *to itself.*

Proof. The lemma is equivalent to this: If $B \subset A \subset S^3$ are unknotted with (A, B) a twisted pair having $N(A, B) \geq 3$, and $h: A \to A$ is a homeomorphism which does not extend to S^3, then $h(B)$ is knotted in S^3. Extendability depends only on the isotopy class of h. Any element of the mapping class group of A (i.e., isotopy classes of homeomorphisms) is represented by a composition of reflections, which extend to S^3, with d^n. Here d is the homeomorphism which, on $S^1 \times D^2$, sends $(\psi, (r, \mu))$ to $(\psi, (r, \mu + \psi))$ (a Dehn twist). Thus we need only show that $d^n(B)$ is knotted in S^3 for $n \neq 0$. If (m, l) is a meridian-longitude pair for A, then, on ∂A, $d^n(m) \sim m$, while $d^n(l) \sim l + n * m$. A core of B is parallel to $b \subset \partial A$, where $b \sim k * l \pm m$. Then $d^n(b) \sim k * d^n(l) \pm d^n(m) = k * l + (n * k \pm 1) * m$. This is knotted unless $|n * k \pm 1| = 1$, which is false since $k \geq 3$ and $n \neq 0$.

For our example, let $A \supset B \supset C$ be solid tori in S^3 so that (A, B) and (B, C) are the same twisted pair with $N(A, B) \geq 3$; so that A and B are unknotted in S^3, but (B, C) is embedded with a single Dehn twist. Now C is knotted in S^3, and we claim that if $f: A \to S^3$ is any unknotted embedding, then $f(C)$ is knotted in S^3. Suppose, to the contrary, that $f: A \to S^3$ is an embedding with both $A' = f(A)$ and $C' = f(C)$ unknotted. By Theorem 1.3, $B' = f(B)$ is also unknotted. By Lemma 4.9, f^{-1} extends to S^3 which

contradicts the fact that $C = f^{-1}(C')$ is knotted. Using the construction in §5 of [10], make a genus one Whitehead manifold O with a prime sequence $\{V_n\}$ so that (V_{3n+3}, V_{3n+2}) is Whitehead's pair, while $V_{3n+2} \supset V_{3n+1} \supset V_{3n}$ is homeomorphic to $A \supset B \supset C$. Then (V_{n+1}, V_n) is unknotted for every n, but (V_{3n+2}, V_{3n}) is knotted for every n. It follows that O cannot be embedded in a compact 3-manifold.

A question, related to the above, that we have not been able to resolve is this: If $\{V_n\}$ is a prime sequence for an irreducible genus one Whitehead manifold O which embeds in a compact 3-manifold, is the homeomorphism type of O determined by the homeomorphism types of the pairs (V_{n+1}, V_n)?

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, DARTMOUTH COLLEGE, HANOVER, NEW HAMPSHIRE 03755